

NOTES FOR THE VANDERBILT SUBFACTOR SEMINAR, FEBRUARY 6TH, 2009. DERIVATIONS ON GROUP-MEASURE SPACE CONSTRUCTIONS

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ABSTRACT. In this talk we will investigate the structure of a class of closable derivations on von Neumann algebras coming from group-measure space constructions. We will then show how to apply these results to obtain new examples of von Neumann algebras which do not arise as group-measure space constructions, for example the von Neumann algebra $L(SL_3(\mathbb{Z}) * G)$ where G is any non-trivial group. Results in greater generality can be found in [27].

Throughout these notes all finite von Neumann algebras will be separable and come with a faithful normal trace which we will denote by τ .

1. THE GROUP-MEASURE SPACE CONSTRUCTION OF MURRAY AND VON NEUMANN [20]

Let Γ be a countable discrete group and suppose we are given a measure preserving action $\sigma : \Gamma \rightarrow \text{Aut}(X, \mu)$ of Γ on a probability space (X, μ) . Consider the abelian von Neumann algebra $A = L^\infty(X, \mu)$ and for each $\gamma \in \Gamma$ consider the mapping (which we still denote by σ_γ) $\sigma_\gamma : A \rightarrow A$ given by $\sigma_\gamma(f)(x) = f(\sigma_{\gamma^{-1}}(x))$, $\forall x \in X$. The assignment $\gamma \mapsto \sigma_\gamma$ defines an action of Γ by integral preserving automorphisms of A .

Let $\mathcal{H} = \ell^2(\Gamma, L^2(X, \mu))$ be the Hilbert space of square summable functions from Γ into $L^2(X, \mu)$. It will be convenient to view vectors in this Hilbert space as formal sums $\xi = \sum_{\gamma \in \Gamma} a_\gamma u_\gamma$ where $a_\gamma \in L^2(X, \mu)$ is the coefficient of the function ξ at γ , i.e. $a_\gamma = \xi(\gamma)$. In this setting the inner product on $\mathcal{H}$ then becomes $\langle \sum_{\gamma \in \Gamma} a_\gamma u_\gamma, \sum_{\lambda \in \Gamma} b_\lambda u_\lambda \rangle = \sum_{\gamma \in \Gamma} \int a_\gamma \bar{b}_\gamma d\mu$.

If $\xi = \sum_{\gamma \in \Gamma} a_\gamma u_\gamma \in \mathcal{H}$, and $\eta = \sum_{\lambda \in \Gamma} b_\lambda u_\lambda \in \mathcal{H}$ we define the convolution of ξ and η to be the formal sum

$$\begin{aligned} \xi \cdot \eta &= (\sum_{\gamma \in \Gamma} a_\gamma u_\gamma) \cdot (\sum_{\lambda \in \Gamma} b_\lambda u_\lambda) \\ &= \sum_{\gamma, \lambda \in \Gamma} a_\gamma \sigma_\gamma(b_\lambda) u_{\gamma\lambda} = \sum_{\gamma \in \Gamma} (\sum_{\lambda \in \Gamma} a_{\gamma\lambda^{-1}} \sigma_{\gamma\lambda^{-1}}(b_\lambda)) u_\gamma. \end{aligned}$$

Note that for each $\gamma \in \Gamma$ we have that

$$\begin{aligned} \|\sum_{\lambda \in \Gamma} a_{\gamma\lambda^{-1}} \sigma_{\gamma\lambda^{-1}}(b_\lambda)\|_1 &\leq \sum_{\lambda \in \Gamma} \|a_{\gamma\lambda^{-1}} \sigma_{\gamma\lambda^{-1}}(b_\lambda)\|_1 \\ &\leq \sum_{\lambda \in \Gamma} \|a_{\gamma\lambda^{-1}}\|_2 \|\sigma_{\gamma\lambda^{-1}}(b_\lambda)\|_2 \\ &\leq (\sum_{\lambda \in \Gamma} \|a_{\gamma\lambda^{-1}}\|_2^2)^{1/2} (\sum_{\lambda \in \Gamma} \|b_\lambda\|_2^2)^{1/2} = \|\xi\|_2 \|\eta\|_2, \end{aligned}$$

hence $\xi \cdot \eta$ is well defined as a function in $\ell^\infty(\Gamma, L^1(X, \mu))$, the space of bounded functions from Γ to $L^1(X, \mu)$.

It may happen that $\xi \cdot \eta$ actually lies in $\mathcal{H} \subset \ell^\infty(\Gamma, L^1(X, \mu))$, for instance this is the case when $\eta \in \mathcal{H}$ is arbitrary and $\xi \in A\Gamma$ the space of functions ξ with finite support and coefficients lying in A . Consider the subset N of $\mathcal{H}$ given by $N = \{x \in \mathcal{H} | x \cdot \eta \in \mathcal{H}, \forall \eta \in \mathcal{H}\}$. Then $A\Gamma \subset N$ and N is an algebra under the multiplication given by convolution, note that N contains a unit u_e . Furthermore we define a $*$ -structure on N by setting

$$(\sum_{\gamma \in \Gamma} a_\gamma u_\gamma)^* = \sum_{\gamma \in \Gamma} \sigma_\gamma(\overline{a_{\gamma^{-1}}}) u_\gamma.$$

We also put a trace on N by the formula

$$\tau(\sum_{\gamma \in \Gamma} a_\gamma u_\gamma) = \int a_e d\mu$$

and in this way N becomes a tracial $*$ -algebra, one checks that N is actually a von Neumann algebra which we will denote by $A \rtimes^\sigma \Gamma$, we will often drop the σ in this notation.

Note that $A \subset A \rtimes \Gamma$ is a von Neumann subalgebra where we identify $a \in A$ with $au_e \in A \rtimes \Gamma$, and our trace τ on $A \rtimes \Gamma$ extends the integral on A . Moreover, given $\gamma \in \Gamma$, u_γ will be a unitary operator which implements the automorphism σ_γ on A , i.e. $u_\gamma a u_\gamma^* = \sigma_\gamma(a)$, $\forall a \in A$. Also note that we may realize N concretely as a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ where the representation is given by convolution (check that convolution describes a closed operator in general and then use the Closed Graph Theorem to conclude that $N \subset \mathcal{B}(\mathcal{H})$).

One example to keep in mind is the case when X is a point so that $A = L^\infty(X, \mu) = \mathbb{C}$. In this case we denote the above construction by $L\Gamma$ and it is called the group von Neumann algebra associated to Γ . $L\Gamma$ will be a II_1 factor if and only if all nontrivial conjugacy classes of Γ are infinite [21], i.e. $|\{\gamma x \gamma^{-1} | \gamma \in \Gamma\}| = \infty, \forall x \in \Gamma, x \neq e$.

On the other hand if we assume that the action is free ($\sigma_\gamma(x) = x, \forall x \in A \implies \gamma = e$) then $A \rtimes \Gamma$ will be a II_1 factor if and only if the action is also ergodic ($\sigma_\gamma(x) = x, \forall \gamma \in \Gamma \implies x \in \mathbb{C}$) [20], and in the case the action is free A is a Cartan subalgebra of $A \rtimes \Gamma$, i.e. A is maximal abelian and its normalizer $\mathcal{N}_{A \rtimes \Gamma}(A) = \{u \in \mathcal{U}(A \rtimes \Gamma) | u A u^* = A\}$ generates $A \rtimes \Gamma$ as a von Neumann algebra.

Remark 1.1. Although group von Neumann algebras arise in this way as a crossed product with $\mathbb{C}$, we will say that a finite von Neumann algebra is a group-measure space construction only if it arises as a crossed product with an abelian von Neumann algebra such that the group is infinite and the action of the group is free.

Group-measure space constructions were the first class of II_1 factors to be introduced [20]. The first examples of II_1 factors which do not arise as a group-space construction are the free group factors $L\mathbb{F}_n$, this is due to Voiculescu in [33]. This has been generalized in [3] to include all amalgamated free products $N = N_1 *_{{N_0}} N_2$ such that N embeds into R^ω , N_0 is finite dimensional and $\dim(N_0)^{-1} - \dim(N_1)^{-1} - \dim(N_2)^{-1} > 0$ (here we use the convention that $\infty^{-1} = 0$).

More recently, in [23] Ozawa and Popa gave a new proof that $L\mathbb{F}_n$ is not a group-measure space construction and they also proved that any non-amenable subfactor of $L\mathbb{F}_n$ is not a group-measure space construction (see also [24]). Houdayer, building on Ozawa and Popa's

results has shown in [15] that $\Gamma = ((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) *_\mathbb{Z} ((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z})$ is an example of a group such that $L\Gamma$ is a factor which is different from a free group factor and yet still has the property that every non-amenable subfactor is not a group-measure space construction. Houdayer also showed that $L(\Gamma_1 * \Gamma_2)$ is not a group-measure space construction whenever $|\Gamma_1| \geq 2$, $|\Gamma_2| \geq 3$, and Γ_1 and Γ_2 are weakly amenable groups with constant 1.

The purpose of this talk is to give another approach to constructing II_1 factors which do not arise as group-measure space constructions. This approach will use deformation/rigidity and intertwining techniques pioneered by Popa and in this way it is much more similar to Ozawa and Popa's approach as opposed to Voiculescu's approach. However it should be pointed out that Ozawa and Popa focus on a class of II_1 factors which have the Haagerup property while we will focus on a class of II_1 factors which do not have the Haagerup property although still have some strong type of deformations and so the results here are mostly disjoint with these earlier results. Another major difference to this approach as opposed to the ones before is that here we will use more directly the group-measure space decomposition and as such these results are more specific to group-measure space constructions rather than von Neumann algebras containing a Cartan subalgebra.

2. THE HAAGERUP PROPERTY AND PROPERTY (T) OF KAZHDAN

Recall that given a countable discrete group Γ , a positive definite function on Γ is a map $\varphi : \Gamma \rightarrow \mathbb{C}$ such that $\forall \sum_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ we have $\sum_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \varphi(\lambda^{-1}\gamma) \geq 0$.

Also recall that given a finite von Neumann algebra N , a completely positive (c.p.) map on N is a map $\Phi : N \rightarrow N$ such that $\forall \sum_n x_n \otimes y_n \in N \otimes_{\text{alg}} N$ we have $\sum_{n,m} \tau(y_m^* \Phi(x_n^* x_n) y_m) \geq 0$. Completely positive maps are automatically $\|\cdot\|_2$ -continuous and hence we may (and often do) view them as bounded operators on $L^2(N, \tau)$.

A positive definite function $\varphi : \Gamma \rightarrow \mathbb{C}$ gives rise to a c.p. map $\Phi_\varphi : L\Gamma \rightarrow L\Gamma$ such that $\forall \sum_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in L\Gamma$ we have:

$$\Phi_\varphi(\sum_{\gamma \in \Gamma} \alpha_\gamma u_\gamma) = \sum_{\gamma \in \Gamma} \alpha_\gamma \varphi(\gamma) u_\gamma.$$

Also, given a c.p. map $\Phi : L\Gamma \rightarrow L\Gamma$ we obtain a positive definite function φ_Φ on Γ by means of the formula:

$$\varphi_\Phi(\gamma) = \tau(\Phi(u_\gamma) u_\gamma^*), \forall \gamma \in \Gamma.$$

One way in which positive definite functions appear is when we have a unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$, together with a vector $\xi \in \mathcal{K}$. Then one can check that the formula given by:

$$\varphi_\xi(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle, \forall \gamma \in \Gamma$$

describes a positive definite function on Γ . It turns out in fact that every positive definite function arises in this way. Specifically, if $\varphi : \Gamma \rightarrow \mathbb{C}$ is a positive definite function then we may place a pseudo-inner product on $\mathbb{C}\Gamma$ by the formula:

$$\langle \sum_{\gamma \in \Gamma} \alpha_\gamma u_\gamma, \sum_{\lambda \in \Gamma} \beta_\lambda u_\lambda \rangle_\varphi = \sum_{\gamma, \lambda \in \Gamma} \overline{\beta_\lambda} \alpha_\gamma \varphi(\lambda^{-1}\gamma).$$

The positivity of this inner product is given by the fact that φ is positive definite. After quotienting by the kernel and completion we obtain a Hilbert space $\mathcal{K}$. Moreover there is a natural unitary representation of Γ on $\mathcal{K}$ given by

$$\pi_\varphi(\gamma_0)\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma = \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_{\gamma_0 \gamma}.$$

One has to check that this action is well defined and preserves the inner product structure, and then the vector $\xi_\varphi = u_e$ gives the formula above. ξ_φ is a cyclic vector to this representation and so we can see that investigating the class of positive definite functions is the same as investigating the class of unitary representations with a fixed cyclic vector.

One may wish to check the correspondence described above satisfies the following relationships:

positive definite functions	pointed unitary representations
$\varphi = 1$	the trivial representation
$\varphi = \delta_e$	the left regular representation on $\ell^2 \Gamma$, $\xi = \delta_e$
$\varphi = \delta_\Lambda$, $\Lambda < \Gamma$	the quasi-regular representation on $\ell^2(\Gamma/\Lambda)$, $\xi = \delta_\Lambda$
φ a character	the one dimensional representation given by the character
addition	direct sum
product	tensor product
pointwise convergence	convergence in the pointed Fell topology

Similar to the group case, for finite von Neumann algebras there is a correspondence between c.p. maps, and Hilbert bimodules. Specifically, to each c.p. map $\Phi : N \rightarrow N$ there exists a unique Hilbert N - N bimodule $\mathcal{H}_\Phi$ and a cyclic vector $\xi_\Phi \in \mathcal{H}_\Phi$ which satisfies the formula:

$$\tau(\Phi(x)y) = \langle x\xi_\Phi y, \xi_\Phi \rangle, \forall x, y \in N.$$

For the specific details of this correspondence we refer the reader to [28] or [29].

One can also check that the correspondence between c.p. maps and pointed Hilbert bimodules satisfies the following relationships:

completely positive maps on N	pointed Hilbert N - N bimodules
$\Phi = \text{id}$	the trivial bimodule $L^2(N, \tau)$ with $\xi = 1$
$\Phi = \tau$	The coarse bimodule $L^2(N, \tau) \bar{\otimes} L^2(N, \tau)$ with $\xi = 1 \otimes 1$
$\Phi = E_B$, $B \subset N$	The basic construction $L^2\langle N, B \rangle$, $\xi = e_B$
$\Phi = \theta$ an automorphism	$L^2(N, \tau)$ with the structure $x \cdot \eta \cdot y = x\eta\theta(y)$, $\xi = 1$
addition	direct sum
composition	Connes fusion
$\ \cdot\ _2$ -pointwise convergence	convergence in the pointed Fell topology

Notions such as amenability, the Haagerup property, or property (T) of Kazhdan may be thought of as measuring how well the trivial representation can be approximated in

other representations, or equivalently from above, how well the constant function 1 can be approximated by other positive definite functions.

Exercise 2.1. Show that a countable discrete group Γ is amenable if and only if there exists a sequence of positive definite functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ which have finite support and converge pointwise to 1.

Definition 2.2 ([14]). A countable discrete group Γ has the Haagerup property if the following equivalent conditions are satisfied:

- (a). There exists a sequence of C_0 positive definite functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ which converge pointwise to 1.
- (b). There exists a C_0 -representation which almost contains invariant vectors.

Some examples of groups with the Haagerup property include amenable groups, free groups, surface groups, Coxeter groups, groups acting properly on trees, and groups acting properly on a space with walls. The Haagerup property is stable under direct product, free product (with amalgamation over a finite subgroup), increasing union, taking subgroups, extensions with amenable quotients, and wreath product. We refer the reader to [4] for more information on the Haagerup property.

The only known obstruction to the Haagerup property is the existence of an infinite subset $X \subset \Gamma$ which has relative property (T) which we define now. This should be compared with Theorem 3.5 below.

Definition 2.3 ([17, 19, 9]). Let Γ be a countable discrete group and $X \subset \Gamma$, the pair (Γ, X) has relative property (T) if the following condition is satisfied:

- (a). Any sequence of positive definite functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ which converges pointwise to 1 also converges uniformly to 1 on X .

Moreover if $X = \Lambda$ is a subgroup then this is equivalent to the following condition:

- (b). Any representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ which does not contain non-zero Λ invariant vectors has a spectral gap, i.e. $\exists F \subset \Gamma$ finite $C_0 > 0$ such that $\|\xi\| \leq C_0 \sum_{\gamma \in F} \|\pi(\gamma)\xi - \xi\|$, $\forall \xi \in \mathcal{K}$.

Γ is said to have property (T) of Kazhdan if the pair (Γ, Γ) has relative property (T).

The first examples of groups with property (T) were given by lattices in $SL_n(\mathbb{R})$, for $n \geq 3$. $(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}), \mathbb{Z}^2)$ is an example of a group-subgroup pair with relative property (T) even though both $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ have the Haagerup property.

Exercise 2.4. Show that a group Γ has the Haagerup property and property (T) if and only if it is finite.

Exercise 2.5. Show that in part (b) of the definition of relative property (T) for subgroups that the set $F \subset \Gamma$ and $C_0 > 0$ may be taken independently of the representation $\mathcal{K}$.

Exercise 2.6. Show that a group Γ with property (T) is finitely generated.

Exercise 2.7. Show that property (T) is stable under taking direct products and quotients.

We also introduce the notions of the Haagerup property and property (T) for finite von Neumann algebras.

Definition 2.8. Let N be a finite von Neumann algebra and $\mathcal{H}$ an N - N Hilbert bimodule. $\mathcal{H}$ is said to be a compact correspondence if it has compact matrix coefficients, i.e. given any bounded vector $\xi \in \mathcal{H}$ the c.p. map $\Phi_\xi : N \rightarrow N$ is compact when viewed as an operator on $L^2(N, \tau)$. Equivalently $\mathcal{H}$ is a compact correspondence if given any two sequences $\{x_n\}_n, \{y_n\}_n \subset (N)_1$ such that $x_n \rightarrow x$ weakly and $y_n \rightarrow y$ weakly, then $\langle x_n \eta_1 y_n, \eta_2 \rangle \rightarrow \langle x \eta_1 y, \eta_2 \rangle$, $\forall \eta_1, \eta_2 \in \mathcal{H}$.

Definition 2.9 ([5]). A finite von Neumann algebra N has the Haagerup property if following equivalent conditions are satisfied:

- (a). There is a sequence of unital, tracial c.p. maps $\Phi_n : N \rightarrow N$ which are compact when viewed as maps on $L^2 N$ and which converge pointwise in $\|\cdot\|_2$ to the identity.
- (b). There is a compact correspondence $\mathcal{H}$ which contains almost central unit vectors.

See [2] for more on compact correspondences and the Haagerup property.

From above relation between positive definite functions and c.p. maps we obtain the following theorem.

Theorem 2.10 ([5]). *A countable discrete group Γ has the Haagerup property if and only if the associated group von Neumann algebra $L\Gamma$ has the Haagerup property.*

Definition 2.11 ([7], [8], [29]). Let N be a finite von Neumann algebra and $B \subset N$ a von Neumann subalgebra then the inclusion $(B \subset N)$ is rigid (or has relative property (T)) if the following equivalent conditions are satisfied:

- (a). Any sequence of unital, tracial c.p. maps $\Phi_n : N \rightarrow N$ which converge pointwise in $\|\cdot\|_2$ to the identity also converges uniformly to the identity on $(B)_1$.
- (b). Any Hilbert bimodule $\mathcal{H}$ which does not contain non-zero B -central vectors must have a spectral gap, i.e. $\exists F \subset N$ finite, $C_0 > 0$ such that $\|\xi\| \leq C_0 \sum_{x \in F} \|x\xi - \xi x\|$.

N has property (T) if the inclusion $(N \subset N)$ is rigid.

The equivalence between property (T) for groups and property (T) for von Neumann algebras is a bit more difficult than for the Haagerup property. One must show that if a unital c.p. map Φ is uniformly close to the identity on $\{u_\gamma\}_{\gamma \in \Gamma}$ then it must be uniformly close to the identity on all of $(L\Gamma)_1$. This is achieved by looking at the pointed Hilbert bimodule $(\mathcal{H}_\Phi, \xi_\Phi)$ associated to Φ and then averaging over $\{u_\gamma\}_{\gamma \in \Gamma}$ to obtain a $L\Gamma$ -central vector which is close to ξ_Φ . This then shows that Φ must be uniformly close to the identity on $(L\Gamma)_1$ and hence we have the following theorem.

Theorem 2.12 ([8]). *A countable discrete group Γ has property (T) if and only if $L\Gamma$ has property (T).*

With the machinery we have developed we will now outline a proof of Connes' result [6] that a II_1 factor with property (T) has countable fundamental group.

Exercise 2.13. Show that a II_1 factor N with property (T) has countable fundamental group by the following steps.

1. Show that an automorphism on a II_1 factor M , $\alpha : M \rightarrow M$ such that $\|\alpha(x) - x\|_2 \leq c < 1$ must be inner. Hint: Average over the unitaries $\alpha(u)u^*$ then take a polar decomposition and show that the partial isometry in the decomposition is actually a unitary which implements α .
2. Show that if M is a II_1 factor with property (T) then the space of inner automorphisms is open (and hence also closed) in the space of all automorphisms under the topology of pointwise $\|\cdot\|_2$ -convergence.
3. Show that if M is a separable II_1 factor then the space of all automorphisms under the topology of pointwise $\|\cdot\|_2$ -convergence is a separable Polish space and so if the space of inner automorphisms is closed then the space of outer automorphisms must be discrete and separable, hence countable.
4. Show that property (T) is closed under taking tensor products.
5. Show that the fundamental group of a II_1 factor N embeds into the outer automorphism group of the tensor product $N \bar{\otimes} N$. Hint: Use the fact that we always have the isomorphism $N \bar{\otimes} N \cong N^t \bar{\otimes} N^{1/t}$, $\forall t > 0$.

3. COCYCLES & CLOSABLE DERIVATIONS

Given a countable discrete group Γ and a unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$, a cocycle is a map $c : \Gamma \rightarrow \mathcal{K}$ which satisfies the identity $c(\gamma_1\gamma_2) = c(\gamma_1) + \pi(\gamma_1)c(\gamma_2)$, $\forall \gamma_1, \gamma_2 \in \Gamma$. A cocycle c is said to be inner if there is a vector $\xi \in \mathcal{K}$ such that $c(\gamma) = \pi(\gamma)\xi - \xi$, $\forall \gamma \in \Gamma$. Cocycles give rise naturally to affine actions on $\mathcal{K}$ by the formula

$$\alpha(\gamma)\xi = \pi(\gamma)\xi + c(\gamma), \forall \gamma \in \Gamma, \xi \in \mathcal{K}.$$

Moreover every affine action of Γ on a Hilbert space $\mathcal{K}$ arises in this way.

Hilbert spaces have the property that any bounded set has a unique Chebyshev center, i.e. if $X \subset \mathcal{K}$ is bounded then there exists a unique point $x_0 \in \mathcal{K}$ which minimizes the value $\sup_{x \in X} \|x - x_0\|$. Affine actions which leave a bounded set invariant will thus leave the Chebyshev center invariant. This gives rise to the following:

Theorem 3.1. *Let $c : \Gamma \rightarrow \mathcal{K}$ be a cocycle and α the associated affine action on $\mathcal{K}$. The following are equivalent.*

- (a). c is inner.
- (b). c is bounded.
- (c). α has bounded orbits.

(d). α has a bounded orbit.

(e). α has a fixed point.

A conditionally negative definite function on Γ is a function $\psi : \Gamma \rightarrow \mathbb{C}$ such that $\forall \sum_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ with $\sum_{\gamma \in \Gamma} \alpha_\gamma = 0$ we have $\sum_{\gamma, \lambda \in \Gamma} \overline{\alpha_{\lambda^{-1}}} \alpha_\gamma \psi(\lambda^{-1}\gamma) \leq 0$. A cocycle $c : \Gamma \rightarrow \mathcal{K}$ gives rise to a positive valued conditionally negative definite function by the formula

$$\psi_c(\gamma) = \|c(\gamma)\|^2, \forall \gamma \in \Gamma.$$

Every non-negative valued conditionally negative definite function arises in this way.

Non-negative valued conditionally negative definite functions are perhaps most useful though for their connection to semigroups of positive definite functions which was established by Schoenberg.

Theorem 3.2 ([32]). *Let $\psi : \Gamma \rightarrow \mathbb{R}$ be a non-negative valued function. Then ψ is conditionally negative definite if and only if $\Phi_t = e^{-t\psi}$ is positive definite $\forall t > 0$ if and only if $\eta_\alpha = \alpha/(\alpha + \psi)$ is positive definite $\forall \alpha > 0$.*

Given a positive definite function $\varphi : \Gamma \rightarrow \mathbb{C}$ it follows that the function $\psi : \Gamma \rightarrow \mathbb{R}$ given by $\psi(\gamma) = \varphi(e)\overline{\varphi(e)} - \varphi(\gamma)\overline{\varphi(\gamma)}$, $\forall \gamma \in \Gamma$ is conditionally negative definite. In particular if we have a sequence of positive definite functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ such that $\lim_{n \rightarrow \infty} |1 - \varphi_n(\gamma)| = 0$, and if we enumerate our group as $\{\gamma_n\}_{n \in \mathbb{N}}$ with $\gamma_1 = e$ then we may take a subsequence φ_{k_n} such that $|1 - \varphi(\gamma_j)\overline{\varphi(\gamma_j)}| < 2^{-n}$, $\forall j \leq n$. We then have that $\psi(\gamma) = \sum_{n \in \mathbb{N}} \varphi_{k_n}(e)\overline{\varphi_{k_n}(e)} - \varphi_{k_n}(\gamma)\overline{\varphi_{k_n}(\gamma)}$ gives a well defined conditionally negative definite function, and moreover ψ will be bounded on a subset $X \subset \Gamma$ if and only if the subsequence φ_{k_n} converges uniformly to 1 on X .

As a consequence we obtain new characterizations of the Haagerup property and property (T).

Theorem 3.3 ([1]). *Let Γ be a countable discrete group, then the following are equivalent:*

- (a). Γ has the Haagerup property.
- (b). There exists a proper conditionally negative definite function on Γ .
- (c). There exists a proper cocycle $c : \Gamma \rightarrow \mathcal{K}$.
- (d). There exists a proper cocycle into a C_0 -representation.

Theorem 3.4 ([11], [13]). *Let Γ be a countable discrete group, then the following are equivalent:*

- (a). Γ has property (T).
- (b). Every conditionally negative definite function on Γ is bounded.
- (c). Every cocycle $c : \Gamma \rightarrow \mathcal{K}$ is inner
- (d). Every affine action of Γ on a Hilbert space has a fixed point.

We also point out an explicit consequence from the proof of Theorem 3.3 which we will use in the sequel.

Theorem 3.5. *Suppose a countable discrete group Γ has a sequence of positive definite functions $\varphi_n : \Gamma \rightarrow \mathbb{C}$ such that $\lim_{n \rightarrow \infty} |1 - \varphi_n(\gamma)| = 0$, $\forall \gamma \in \Gamma$ and such that given any sequence $m_k \in \mathbb{N}$ there exists a sequence $n_k \in \mathbb{N}$ with $n_k \geq m_k$, $\forall k \in \mathbb{N}$ and φ_{n_k} does not converge uniformly on any infinite subset, i.e. $\forall X \subset \Gamma$ infinite $\limsup_{k \rightarrow \infty} \sup_{\gamma \in X} |1 - \varphi_{n_k}(\gamma)| \neq 0$. Then Γ has the Haagerup property.*

Proof. Note that by considering the positive definite functions $\gamma \mapsto |\Re(\varphi_n(\gamma)/\varphi_n(e))|^2$ we may assume that φ_n takes real values between 0 and 1. Let $\{\gamma_n\}_n$ be an enumeration of Γ , since $\lim_{n \rightarrow \infty} |1 - \varphi_n(\gamma)| = 0$, $\forall \gamma \in \Gamma$ we may construct a sequence m_k such that $\forall n \geq m_k$ we have $1 - \varphi_n(\gamma_j) < 4^{-k}$, $\forall 1 \leq j \leq k$.

By assumption we may then find a sequence n_k such that $n_k \geq m_k$ and φ_{n_k} does not converge uniformly on any infinite subset of Γ . Define the conditionally completely negative function ψ by $\psi(\gamma) = \sum_{k=1}^{\infty} 2^k(1 - \varphi_{n_k}(\gamma))$. Note that since $n_k \geq m_k$, we have $\psi(\gamma) < \infty$, $\forall \gamma \in \Gamma$, and if ψ is bounded by K on a set $X \subset \Gamma$ then $\forall x \in X$ we have $1 - \varphi_{n_k}(x) \leq K2^{-k}$, $\forall k \in \mathbb{N}$ hence X must be finite and so ψ is proper. $\square$

Exercise 3.6. Show that if $\psi : \Gamma \rightarrow \mathbb{R}$ is a non-negative valued conditionally negative definite function then ψ^β is also conditionally negative definite $\forall 0 < \beta \leq 1$. Hint: Show that ψ^β is the limit of bounded conditionally negative definite functions by using the formula:

$$s^\beta = \frac{\sin((1-\beta)\pi)}{\pi} \int_0^\infty \frac{s}{t+s} t^{\beta-1} dt.$$

The analogue of a cocycle in finite von Neumann algebras is the notion of a closable derivation. We briefly sketch the basic theory of closable derivations here.

Suppose N is a finite von Neumann algebra, $D(\delta) \subset N$ a weakly dense $*$ -subalgebra, $\mathcal{H}$ an N - N Hilbert bimodule, and $\delta : D(\delta) \rightarrow \mathcal{H}$ a derivation ($\delta(xy) = x\delta(y) + \delta(x)y$, $\forall x, y \in D(\delta)$), which is closable (as an unbounded operator from $L^2(N, \tau)$ to $\mathcal{H}$), and real ($\langle \delta(x), y\delta(z) \rangle = \langle \delta(z^*)y^*, \delta(x^*) \rangle$, $\forall x, y, z \in D(\delta)$).

It follows from [30] and [12] that $D(\bar{\delta}) \cap N$ is a $*$ -subalgebra of N and $\bar{\delta}|_{D(\bar{\delta}) \cap N}$ is again a derivation. Let $\Delta = \delta^* \bar{\delta}$, then Δ is the generator of a completely Dirichlet form [30]. Associated to Δ are two natural deformations of N , the first is the completely positive semigroup (completely Markovian semigroup) $\{\Phi_t\}_{t>0}$, each $\Phi_t = \exp(-t\Delta)$ is a c.p. map which is unital ($\Phi_t(1) = 1$), tracial ($\tau \circ \Phi_t = \tau$), and positive ($\tau(\Phi_t(x)x^*) \geq 0$, $\forall x \in N$), moreover the semigroup property is satisfied ($\Phi_{t+s} = \Phi_t \circ \Phi_s$, $\forall s, t > 0$), and $\forall x \in N$, $\|x - \Phi_t(x)\|_2 \rightarrow 0$, as $t \rightarrow 0$. The second deformation associated to Δ is the deformation coming from resolvent maps $\{\eta_\alpha\}_{\alpha>0}$, again each $\eta_\alpha = \alpha(\alpha + \Delta)^{-1}$ is a unital, tracial, positive, c.p. map such that $\forall x \in N$, $\|x - \eta_\alpha(x)\|_2 \rightarrow 0$, as $\alpha \rightarrow \infty$, furthermore $\beta\eta_\alpha - \alpha\eta_\beta = (\beta - \alpha)\eta_\alpha \circ \eta_\beta$, $\forall \alpha, \beta > 0$.

The relationship between these maps are as follows and can be found for example in [18]:

$$\Delta = \lim_{t \rightarrow 0} \frac{1}{t}(\text{id} - \Phi_t) = \alpha(\eta_\alpha^{-1} - \text{id}) = \lim_{\alpha \rightarrow \infty} \alpha(\text{id} - \eta_\alpha),$$

$$\Phi_t = \exp(-t\Delta) = \lim_{\alpha \rightarrow \infty} \exp(-t\alpha(\text{id} - \eta_\alpha)),$$

$$\eta_\alpha = \alpha(\alpha + \Delta)^{-1} = \alpha \int_0^\infty e^{-\alpha t} \Phi_t dt.$$

Note that we will use the same symbols Δ , Φ_t , and η_α for the maps on N as well as the corresponding extensions to $L^2(N, \tau)$. Also note that η_α maps into the domain of Δ and $\Delta \circ \eta_\alpha = \alpha(\text{id} - \eta_\alpha)$. Furthermore we have that $\text{Range}(\eta_\alpha) = D(\Delta) \subset D(\bar{\delta})$, $D(\Delta^{\frac{1}{2}}) = D(\bar{\delta}) = \text{Range}(\eta_\alpha^{1/2})$ and $\forall x \in D(\bar{\delta})$, $\|\Delta^{\frac{1}{2}}(x)\|_2 = \|\delta(x)\|_2$.

From the point of view of uniform convergence the deformations $\{\Phi_t\}_{t \rightarrow 0}$ and $\{\eta_\alpha\}_{\alpha \rightarrow \infty}$ are equivalent in the sense that if $B \subset N$ is a von Neumann subalgebra then one will converge uniformly on $(B)_1$ if and only if the other does also.

We mention that $\Delta^{\frac{1}{2}}$ also generates a completely positive deformation as is shown in [31] by the formula: $\Delta^{\frac{1}{2}} = \pi^{-1} \int_0^\infty t^{-1/2}(\text{id} - \eta_t) dt$.

Example 3.7. Suppose Γ is a countable discrete group, $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$ is an orthogonal representation, and $c : \Gamma \rightarrow \mathcal{K}$ is a 1-cocycle. Then as we showed above we have associated to this cocycle a conditionally negative definite function ψ given by $\psi(\gamma) = \|c(\gamma)\|_2^2$, there is also a semigroup of positive definite functions $\{\varphi_t\}_t$ given by $\varphi_t(\gamma) = e^{-t\psi(\gamma)}$, and the set of positive definite resolvents $\{\chi_\alpha\}_\alpha$ given by $\chi_\alpha(\gamma) = \alpha/(\alpha + \psi(\gamma))$.

Let $\mathcal{H} = \mathcal{K} \bar{\otimes}_{\mathbb{R}} L^2(L\Gamma)$ and equip $\mathcal{H}$ with the $L\Gamma$ bimodule structure which satisfies $u_\gamma(\xi \otimes \xi') = \pi(\gamma)\xi \otimes u_\gamma \xi'$ and $(\xi \otimes \xi')u_\gamma = \xi \otimes \xi' u_\gamma$, $\forall \gamma \in \Gamma$, $\xi \in \mathcal{H}$, $\xi' \in L^2(L\Gamma)$. Let $\delta_c : \mathbb{C}\Gamma \rightarrow \mathcal{H}$ be the derivation which satisfies $\delta_c(u_\gamma) = c(\gamma) \otimes u_\gamma$, $\forall \gamma \in \Gamma$, then δ_c is a real closable derivation and so as described above we can associated with δ_c the c.c.n. map Δ along with the deformations $\{\Phi_t\}_t$ and $\{\eta_\alpha\}_\alpha$. It can be easily checked that we have the following relationships:

$$\Delta(u_\gamma) = \psi(\gamma)u_\gamma, \forall \gamma \in \Gamma,$$

$$\Phi_t(u_\gamma) = \varphi_t(\gamma)u_\gamma, \forall \gamma \in \Gamma, t > 0,$$

$$\eta_\alpha(u_\gamma) = \chi_\alpha(\gamma)u_\gamma, \forall \gamma \in \Gamma, \alpha > 0.$$

Note that in this case we have that if $\Lambda < \Gamma$ then the derivation $\delta_{c|_{\mathbb{C}\Lambda}}$ is inner if and only if the cocycle $c|_\Lambda$ is inner if and only if the deformation $\{\eta_\alpha\}_\alpha$ converges uniformly on $(L\Lambda)_1$. Note also that if $\mathcal{K}$ is the left regular representation of Γ then $\mathcal{H}$ is the coarse correspondence for $L\Gamma$. Also if $\mathcal{K}$ is a C_0 -representation then $\mathcal{H}$ is a compact correspondence.

Example 3.8. Suppose (M_1, τ_1) and (M_2, τ_2) are finite diffuse von Neumann algebras, and let $(M, \tau) = (M_1 * M_2, \tau_1 * \tau_2)$. If we let $\delta_i : M_1 *_{\text{Alg}} M_2 \rightarrow L^2(M) \otimes L^2(M)$ be the unique derivation which satisfies $\delta_i(x) = x \otimes 1 - 1 \otimes x$, $\forall x \in M_i$ and $\delta_i(y) = 0$, $\forall y \in M_j$ where $j \neq i$. Then it is easy to check that δ_i defines a closable real derivation and a simple calculation (see for example Corollary 4.2 and the following remark in [25]) shows that the associated semigroups of c.p. maps are given by $\Phi_s^1 = (e^{-2s}\text{id} + (1 - e^{-2s})\tau) * \text{id}$, and $\Phi_s^2 = \text{id} * (e^{-2s}\text{id} + (1 - e^{-2s})\tau)$. In particular we have that $\{\Phi_s^j\}_s$ does not converge uniformly on $(M)_1$ as $s \rightarrow 0$.

Since the range of $\eta_\alpha^{1/2}$ is the same as the domain of δ we may take the composition $\delta \circ \eta_\alpha^{1/2}$ to obtain a bounded operator from $L^2(N, \tau)$ to $\mathcal{H}$ whose norm is no more than $(2\alpha)^{1/2}$. In fact $\alpha\|x - \eta_\alpha(x)\|_2^2 \leq \|\delta \circ \eta_\alpha^{1/2}(x)\|_2^2 = \alpha\tau((x - \eta_\alpha(x))x^*) \leq \alpha\|x - \eta_\alpha(x)\|_2, \forall x \in N$. It will be convenient therefore to use the following notation, we will let $\zeta_\alpha = \eta_\alpha^{1/2}$, and we will let $\tilde{\delta}_\alpha = \alpha^{-1/2}\delta \circ \zeta_\alpha$. The main technical lemma we will use regarding closable derivations is that $\tilde{\delta}_\alpha$ is sufficiently close to δ to remember the bimodule structure. Specifically we have the following estimate for which a proof can be found in [26] or [24]. We note that in the following estimate the term $\tilde{\delta}_\alpha(a)$ will be uniformly small for $a \in F$ and thus we may omit this term from the inequality. We have chosen to keep this term however in order to emphasize the fact that $\tilde{\delta}_\alpha$ is almost a derivation.

Lemma 3.9 ([26]). *Using the same notation as above if $F \subset (N)_1$, such that $\{\eta_\alpha\}_\alpha$ converges uniformly on F (F possibly infinite), then $\forall \varepsilon > 0, \exists \alpha_0 > 0$, such that $\forall \alpha \geq \alpha_0$ we have that $\|\tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x)\|_2^2 < \varepsilon$, and $\|\tilde{\delta}_\alpha(xa) - \tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \zeta_\alpha(x)\tilde{\delta}_\alpha(a)\|_2^2 < \varepsilon$, $\forall a \in F, x \in (N)_1$.*

Specifically it follows from [24] that the left hand sides of the equation above will be less than or equal to $10\|\tilde{\delta}_\alpha(x)\|$.

4. II_1 FACTORS NOT ARISING AS GROUP-MEASURE SPACE CONSTRUCTIONS

Definition 4.1. Let $\mathcal{C}$ be the class of groups Γ such that Γ does not have the Haagerup property and there exists an unbounded cocycle into a C_0 -representation.

Examples of groups in $\mathcal{C}$ include all free products which do not have the Haagerup property or more generally all groups Γ not having the Haagerup property such that $\beta_1^{(2)}(\Gamma) > 0$. Examples of groups which are not in the class $\mathcal{C}$ (in addition to the ones with Haagerup property of course) include all groups which are a direct product of two infinite groups.

The class $\mathcal{C}$ also does not contain all groups which contain an infinite normal abelian subgroup, this follows from [10] and for comparison to the von Neumann algebra results below we will repeat the argument here.

Theorem 4.2 ([10]). *Suppose Γ is a countable discrete group with $\Lambda \triangleleft \Gamma$ an infinite abelian normal subgroup. Then any cocycle $c : \Gamma \rightarrow \mathcal{K}$ into a C_0 -representation must be either bounded or proper. In particular if Γ does not have the Haagerup property then all such cocycles must be bounded (and hence inner).*

Proof. Let $c : \Gamma \rightarrow \mathcal{K}$ be such a cocycle. If c is not proper then there exists an infinite sequence $\gamma_n \rightarrow \infty$ in Γ such that c is bounded by K on $\{\gamma_n\}_n$. Take an element $a \in \Lambda$, then by the cocycle identity c is bounded by $3K$ on the set $\{\gamma_n a \gamma_n^{-1}\}_n \subset \Lambda$. If this subset is infinite for some a then take a subsequence $\{a_n = \gamma_{k_n} a \gamma_{k_n}^{-1}\}_n$ which goes to infinity. Then given any $b \in \Lambda$ we have $2\|c(b)\|^2 = \lim_{n \rightarrow \infty} \|\pi(a_n)c(b) - c(b)\|^2$ from the fact that the representation is C_0 . On the other hand by using the cocycle identity we have

$$\lim_{n \rightarrow \infty} \|\pi(a_n)c(b) - c(b)\|^2 = \lim_{n \rightarrow \infty} \|c(a_n b) - c(a_n) - c(b)\|^2$$

$$= \lim_{n \rightarrow \infty} \|\pi(b)c(a_n) - c(a_n)\|^2 \leq (6K)^2.$$

If $\{\gamma_n a \gamma_n^{-1}\}_n$ is finite then there is a subsequence $\{\gamma_{k_n}\}_n$ such that $[\gamma_{k_n}^{-1} \gamma_{k_n}, a] = e$, $\forall n, m \in \mathbb{N}$. Thus if $\{\gamma_n a \gamma_n^{-1}\}_n$ is finite $\forall a \in \Lambda$ then by taking a diagonal subsequence we then construct in this way a sequence γ'_n such that $\gamma'_n \rightarrow \infty$, $[\gamma'_n, a] \rightarrow e$, $\forall a \in \Lambda$, and $\|c(\gamma'_n)\| < 2K$. Then using the same estimate as above we have that $\forall a \in \Lambda$

$$\begin{aligned} 2\|c(b)\|^2 &= \lim_{n \rightarrow \infty} \|\pi(\gamma'_n)c(a) - c(a)\|^2 \\ &= \lim_{n \rightarrow \infty} \|\pi(a)c(\gamma'_n) - c(\gamma'_n)\|^2 \leq (4K)^2. \end{aligned}$$

Hence in either case we have that $\|c(a)\| \leq 6K$, $\forall a \in \Lambda$. Now take $\gamma \in \Gamma$, and take $a_n \in \Lambda$, $a_n \rightarrow \infty$. Then

$$\begin{aligned} \|c(\gamma)\|^2 &= \lim_{n \rightarrow \infty} \|\pi(a_n)c(\gamma) - c(\gamma)\|^2 \\ &= \lim_{n \rightarrow \infty} \|c(a_n\gamma) - c(a_n) - c(\gamma)\|^2 \\ &= \lim_{n \rightarrow \infty} \|\pi(\gamma)c(\gamma^{-1}a_n\gamma) - c(a_n)\|^2 \leq (12K)^2. \end{aligned}$$

Thus we have shown that $\|c(\gamma)\| \leq 12K$, $\forall \gamma \in \Gamma$. $\square$

We now prove two theorems for von Neumann algebras which may be thought of as analogues to the previous theorem for groups.

Theorem 4.3. *Let M be a II_1 factor and $A \subset M$ a maximal abelian von Neumann subalgebra. Suppose that $\delta : M \rightarrow \mathcal{H}$ is a closable real derivation with associated deformation η_α and such that $\mathcal{H}$ is a compact A - A correspondence. If there is a non- $\|\cdot\|_2$ -precompact subset $X \subset \mathcal{U}(A)$ on which the deformation η_α converges uniformly then the deformation η_α converges uniformly on all of $(A)_1$.*

Proof. Using the same notation as above, suppose that $X \subset \mathcal{U}(A)$ is non- $\|\cdot\|_2$ -precompact and suppose η_α converges uniformly on X . By taking a countable subset of X with the same property we will assume that we can enumerate X as $X = \{v_n\}_n$.

Since $\{v_n\}_n$ is not $\|\cdot\|_2$ -precompact it follows that there is a subsequence such that no finer subsequence converges in $\|\cdot\|_2$. Using the fact that the unit ball is weakly compact we may then take another subsequence which weakly converges to an element $x \in (A)_1$. Hence we will assume that v_n converges weakly to $x \in (A)_1$ but v_n does not converges in $\|\cdot\|_2$.

If x were a unitary then we would have $\|x - v_n\|_2^2 = 2 - \tau(xv_n^*) - \tau(x^*v_n) \rightarrow 0$ and hence since we are assuming that we do not have $\|\cdot\|_2$ convergence we conclude that x is not a unitary. In particular this means that we can take a non-zero spectral project p of $|x|$ so that $\|px\|_\infty = c < 1$.

Now let $\varepsilon > 0$ be given. Take $\alpha_0 > 0$ such that $\forall \alpha > \alpha_0$ we have $\|\tilde{\delta}_\alpha(v_n)\|_2 < (1 - c)\varepsilon/20$, $\forall n \in \mathbb{N}$, and $\|\tilde{\delta}_\alpha(p)\| < (1 - c)\varepsilon/20$. Given $a \in (A)_1$ we then have

$$\begin{aligned} \|\tilde{\delta}_\alpha(pa)\|_2^2 &= |\langle \tilde{\delta}_\alpha(v_n p a v_n^*), \tilde{\delta}_\alpha(pa) \rangle| \\ &\leq |\langle \zeta_\alpha(v_n p) \tilde{\delta}_\alpha(pa) \zeta_\alpha(v_n), \tilde{\delta}_\alpha(pa) \rangle| + (1 - c)\varepsilon \end{aligned}$$

$$\begin{aligned} & \rightarrow_{n \rightarrow \infty} |\langle \zeta_\alpha(xp) \tilde{\delta}_\alpha(pa) \zeta_\alpha(x), \tilde{\delta}_\alpha(pa) \rangle| + (1 - c)\varepsilon \\ & \leq \|xp\|_\infty \|\tilde{\delta}_\alpha(a)\|_2^2 + (1 - c)\varepsilon. \end{aligned}$$

Hence $\|\tilde{\delta}_\alpha(pa)\|_2^2 \leq \varepsilon$, $\forall a \in (A)_1$.

Thus η_α converges uniformly on $p(A)_1$. If by Zorn's Lemma we denote by $p_{\max}$ the maximal projection $q \in \mathcal{P}(A)$ such that η_α converges uniformly on $q(A)_1$ then it is easy to see that $p_{\max}$ must live in the center of N , on the other hand we showed above that $p_{\max} \geq p \neq 0$ and hence by factoriality we must have that $p_{\max} = 1$, i.e. η_α converges uniformly on $(A)_1$. $\square$

Theorem 4.4. *Let Γ be a countable discrete group, and $\sigma : \Gamma \rightarrow \text{Aut}(A)$ a free ergodic action. Let $N = A \rtimes \Gamma$ be the resulting group-measure space construction. Suppose $N \subset M$ and there exists a closable real derivation $\delta : M \rightarrow \mathcal{H}$ such that $\mathcal{H}$ is a compact N - N bimodule and the deformation η_α does not converge uniformly on $(N)_1$. Then Γ has the Haagerup property.*

Proof. We will show that given an infinite subset $X \subset \Gamma$ the deformation does not converges uniformly on X , the fact that Γ has the Haagerup property then follows from Theorem 3.5.

Suppose by contradiction that η_α did converge uniformly on an infinite set $\{\gamma_n\}_n \subset \Gamma$. If there exists some $a \in \mathcal{U}(A)$ such that $\{u_{\gamma_n} a u_{\gamma_n}^*\}_n$ is not $\|\cdot\|_2$ -precompact then from Theorem 4.3 we would have that η_α converges uniformly on $\mathcal{U}(A)$ and hence. On the other hand if $\{u_{\gamma_n} a u_{\gamma_n}^*\}_n$ is $\|\cdot\|_2$ -precompact, $\forall a \in \mathcal{U}(A)$ then given any $a \in \mathcal{U}(A)$, $\alpha > 0$ we have that $\{\tilde{\delta}_\alpha(u_{\gamma_n} a u_{\gamma_n}^*)\}_n$ is also precompact and hence

$$\begin{aligned} \|\tilde{\delta}_\alpha(a)\|_2^2 & \sim |\langle \zeta_\alpha(u_{\gamma_n}) \tilde{\delta}_\alpha(a) \zeta_\alpha(u_{\gamma_n}^*), \zeta_\alpha(u_{\gamma_n}) \tilde{\delta}_\alpha(a) \zeta_\alpha(u_{\gamma_n}^*) \rangle| \\ & \sim |\langle \zeta_\alpha(u_{\gamma_n}) \tilde{\delta}_\alpha(a) \zeta_\alpha(u_{\gamma_n}^*), \tilde{\delta}_\alpha(u_{\gamma_n} a u_{\gamma_n}^*) \rangle| \rightarrow_{n \rightarrow \infty} 0. \end{aligned}$$

Thus again we show that η_α converges uniformly on $\mathcal{U}(A)$.

However if η_α converges uniformly on $\mathcal{U}(A)$ then it follows from Theorem 4.5 in [26] that η_α must also converge uniformly on $\mathcal{N}_N(A)'' = N$ which contradicts our assumption on the deformation. $\square$

We mention one immediate consequence which follows from [26].

Corollary 4.5. *Let Γ be a countable discrete group, and $\sigma : \Gamma \rightarrow \text{Aut}(A)$ a free ergodic action. Let $N = A \rtimes \Gamma$ be the resulting group-measure space construction. If Γ does not have the Haagerup property then N is L^2 -rigid, in particular N satisfies the Kurosh Theorem ([22], [16]) for free products.*

To obtain new examples of groups which do not arise as group-measure space constructions we first recall a result of Popa:

Theorem 4.6 ([29]). *Let Γ be a countable discrete group, and $\sigma : \Gamma \rightarrow \text{Aut}(A)$ a free ergodic action. Let $N = A \rtimes \Gamma$ be the resulting group-measure space construction. If Γ has the Haagerup property and $B \subset N$ is a von Neumann subalgebra such that $(B \subset N)$ is rigid, then there exists a corner of B which embeds into A inside of N , i.e. there exists a non-zero projection f in $B' \cap \langle N, A \rangle$ of finite trace.*

We remark that if $B \subset N$ is a type II_1 von Neumann subalgebra and $A \subset N$ is a type I subalgebra then we cannot have that a corner of B embeds into A inside of N . Thus it follows that if Γ has the Haagerup property and $\sigma : \Gamma \rightarrow \text{Aut}(A)$ is a free ergodic action then $N = A \rtimes \Gamma$ cannot contain a type II_1 subalgebra B such that $(B \subset N)$ is rigid. As a consequence from Theorems 4.4 and 4.6 we obtain the following:

Corollary 4.7. *Suppose $N = L\Gamma$ where $\Gamma \in \mathcal{C}$ is an i.c.c. group or suppose $N = N_1 * N_2$ where N_j are diffuse finite von Neumann algebras. If N contains a type II_1 rigid subalgebra then N is not a group-measure space construction.*

A specific example of a group which satisfies the hypotheses of the previous corollary is the group $SL_3(\mathbb{Z}) * G$ where G is any non-trivial group.

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